Asymmetric Christmas Trees*

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Abstract

Numbers sometimes produce extraordinary patterns that typically conceal very simple and elegant rules. The numbers described in this note are integers, but by looking more closely at the underlying regularity one can envisage extending the idea to real numbers as well. On the one hand, interpretations related to the representation of integers emerge; on the other, the Christmas trees illustrated below can also be interpreted in terms of the dynamics of discrete-time linear systems. In any case, their asymmetry makes sense only provided that only if the left part and the right part are balanced so that the tree does not fall.

1 Introduction

Francesco Vannucci, a dear friend from Genoa, sent me a curious Christmas tree. It is made of numbers, with pleasing shapes that reveal the elegance of numbers. Patterns of this kind belong to a long tradition of recreational mathematics, where numerical representations often give rise to unexpected visual regularities [1, 2].

The tree is asymmetrical, but very beautiful, and it is this one:

$$0 \cdot 9 + 8 = 8$$
$$9 \cdot 9 + 7 = 88$$
$$98 \cdot 9 + 6 = 888$$
$$987 \cdot 9 + 5 = 8888$$
$$9876 \cdot 9 + 4 = 88888$$
$$98765 \cdot 9 + 3 = 888888$$
$$987654 \cdot 9 + 2 = 8888888$$
$$9876543 \cdot 9 + 1 = 88888888$$
$$98765432 \cdot 9 + 0 = 888888888$$
$$987654321 \cdot 9 - 1 = 8888888888$$

But where does this strange pattern come from? Interestingly, this nicely arises from expressing integer in any base by using positional representation [5]. I began thinking in different

*This article is dedicated to the memory of Giovanni Soda.

bases to build different trees, For $b = 8$ I came up with the following similar Christmas tree

$$\begin{aligned}
0 \cdot 7 + 6 &= 6 \\
7 \cdot 7 + 5 &= 66 \\
76 \cdot 7 + 4 &= 666 \\
765 \cdot 7 + 3 &= 6666 \\
7654 \cdot 7 + 2 &= 66666 \\
76543 \cdot 7 + 1 &= 666666 \\
765432 \cdot 7 + 0 &= 6666666 \\
7654321 \cdot 7 - 1 &= 66666666,
\end{aligned}$$

Here all numbers have to be interpreted as numbers in base $b = 8$ instead of the ordinary $b = 10$. Clearly bigger trees require a high base b. For example, for $b = 16$ the tree becomes

$$\begin{aligned}
0 \cdot F + E &= E \\
F \cdot F + D &= EE \\
FE \cdot F + C &= EEE \\
FED \cdot F + B &= EEEE \\
FEDC \cdot F + A &= EEEEE \\
FEDCB \cdot F + 9 &= EEEEEE \\
FEDCBA \cdot F + 8 &= EEEEEEE \\
FEDCBA9 \cdot F + 7 &= EEEEEEEE \\
FEDCBA98 \cdot F + 6 &= EEEEEEEEE \\
FEDCBA987 \cdot F + 5 &= EEEEEEEEEE \\
FEDCBA9876 \cdot F + 4 &= EEEEEEEEEEE \\
FEDCBA98765 \cdot F + 3 &= EEEEEEEEEEEE \\
FEDCBA987654 \cdot F + 2 &= EEEEEEEEEEEEE \\
FEDCBA9876543 \cdot F + 1 &= EEEEEEEEEEEEEE \\
FEDCBA98765432 \cdot F + 0 &= EEEEEEEEEEEEEEE \\
FEDCBA987654321 \cdot F - 1 &= EEEEEEEEEEEEEEEE
\end{aligned}$$

One could also make other light bulbs and Christmas decorations, provided they suggest an order like numbers do, and then build other trees with them. I think it would be beautiful, but for now I use only the ordinary symbols of numbers.

2 Christmas tree and codes

We can interpret the above property in terms of positional representation of number in base $b > 1$. The left and the right parts of the Christmas tree can be written as

$$\begin{aligned}
L(n) &= (b-1)\sum_{k=1}^{n-1}(b-\kappa)b^{n-\kappa-1} + b - n - 1 \\
R(n) &= \frac{b-2}{b-1}(b^n - 1)
\end{aligned}$$

	n	$L(n)$	$R(n)$
	1	$0 \cdot 9 + 8$	8
	2	$9 \cdot 9 + 7$	88
	3	$98 \cdot 9 + 6$	888
	4	$987 \cdot 9 + 5$	8888
$b = 10:$	5	$9876 \cdot 9 + 4$	88888
	6	$98765 \cdot 9 + 3$	888888
	7	$987654 \cdot 9 + 2$	8888888
	8	$9876543 \cdot 9 + 1$	88888888
	9	$98765432 \cdot 9 + 0$	888888888
	10	$987654321 \cdot 9 - 1$	8888888888

Table 1: The Christmas tree pattern, in which balance is maintained between the left $L(n)$ and right sides $R(n)$.

where $n \in [1, b]$. The beauty of the numbers can be expressed in terms of balance of the left and right side of the Christmas tree.

An interesting structural perspective emerges when observing the individual digits in $L(n)$. Each digit of the left-hand expression is of the form $b-k$, for $k = 1, \ldots, n$, forming a descending sequence of complements relative to the base b. The right-hand side, in contrast, collapses this varying sequence into a uniform string of the digit $b-2$. In this sense, the tree encodes a combinatorial compression: the same integer can be represented either as a highly structured sequence of complementary digits or as a constant repeated symbol, highlighting an intrinsic balance between variation and uniformity in positional number systems.

The balance of the tree is established by the following proposition.

Proposition 1.

$$L(n) = (b-1)\sum_{k=1}^{n-1}(b-k)b^{n-k-1} + b - n - 1 = R(n) = \frac{b-2}{b-1}(b^n - 1).$$

Proof. The proof undergoes the four following steps.
Step 1: Change the summation index. Let $j = n - k$. Then when $k = 1, j = n - 1$ and when $k = n-1, j = 1$. Also, $b - k = b - (n - j) = b - n + j$ and $b^{n-k-1} = b^{j-1}$. So the sum becomes

$$\sum_{j=1}^{n-1}(b - n + j)b^{j-1}.$$

Step 2: Split the sum.

$$\sum_{j=1}^{n-1}(b-n+j)b^{j-1} = \sum_{j=1}^{n-1}(b-n)b^{j-1} + \sum_{j=1}^{n-1} jb^{j-1}.$$

Thus,

$$L(n) = (b-1)\left[\sum_{j=1}^{n-1}(b-n)b^{j-1} + \sum_{j=1}^{n-1} jb^{j-1}\right] + b - n - 1.$$

Step 3: Evaluate the geometric sum.

$$\sum_{j=1}^{n-1} b^{j-1} = 1 + b + \cdots + b^{n-2} = \frac{b^{n-1}-1}{b-1}.$$

So the first term becomes

$$(b-1)(b-n)\sum_{j=1}^{n-1} b^{j-1} = (b-n)(b^{n-1}-1).$$

Step 4: Evaluate the weighted sum. Using the formula

$$\sum_{j=1}^{m} jr^{j-1} = \frac{1-r^m}{(1-r)^2} - \frac{mr^m}{1-r}, \quad r \neq 1,$$

we get

$$\sum_{j=1}^{n-1} jb^{j-1} = \frac{1-b^{n-1}}{(b-1)^2} + \frac{(n-1)b^{n-1}}{b-1}.$$

Multiply by $b-1$:

$$(b-1)\sum_{j=1}^{n-1} jb^{j-1} = \frac{1-b^{n-1}}{b-1} + (n-1)b^{n-1}.$$

Step 5: Combine all terms.

$$\begin{aligned}
L(n) &= (b-n)(b^{n-1}-1) + (n-1)b^{n-1} + \frac{1-b^{n-1}}{b-1} + b - n - 1 \\
&= (b-1)b^{n-1} - \frac{b^{n-1}}{b-1} - 1 + \frac{1}{b-1} + \dots \\
&= \frac{b(b-2)b^{n-1}}{b-1} - \frac{b-2}{b-1} \\
&= \frac{b-2}{b-1}(b^n - 1) \\
&= R(n).
\end{aligned}$$

□ □

Another nice proof follows the classic telescopic trick.

Telescopic Proof:

Telescoping arguments of this type are classical and appear frequently in summation theory [3].
Step 1: Split the summand as

$$(b-k)b^{n-k-1} = b^{n-k} - kb^{n-k-1}.$$

So

$$\sum_{k=1}^{n-1}(b-k)b^{n-k-1} = \sum_{k=1}^{n-1} b^{n-k} - \sum_{k=1}^{n-1} kb^{n-k-1}.$$

Step 2: Change indices in the second sum. Let $i = n-k$, then $k = n-i$ and $i = 1, \dots, n-1$:

$$\sum_{k=1}^{n-1} kb^{n-k-1} = \sum_{i=1}^{n-1}(n-i)b^{i-1} = \sum_{i=1}^{n-1} nb^{i-1} - \sum_{i=1}^{n-1} ib^{i-1}.$$

Hence,

$$\sum_{k=1}^{n-1}(b-k)b^{n-k-1} = \sum_{i=1}^{n-1} b^i - \sum_{i=1}^{n-1} nb^{i-1} + \sum_{i=1}^{n-1} ib^{i-1} = \sum_{i=1}^{n-1} \left(b^i - nb^{i-1} + ib^{i-1}\right).$$

Combine terms:

$$b^i - nb^{i-1} + ib^{i-1} = b^{i-1}(b+i-n) = b^{i-1}((b-1)-(n-1-i)).$$

This is now a telescopic sum in i.

Step 3: Sum the telescopic series:

$$\sum_{i=1}^{n-1}(b-k)b^{n-k-1} = \frac{b^n - b(n-1) - 1}{b-1}.$$

Step 4: Multiply by $b-1$ and add the remaining constant $b-n-1$:

$$L(n) = (b-1)\sum_{k=1}^{n-1}(b-k)b^{n-k-1} + b - n - 1 = b^n - b(n-1) - 1 + b - n - 1 = \frac{b-2}{b-1}(b^n - 1).$$

Then $L(n) = R(n)$

From another viewpoint, the telescopic cancellation underlying the proof can be interpreted as a discrete analogue of integration by parts. Here, the exponential weights b^{n-k-1} act as a discrete kernel, while the linearly decreasing coefficients $(b-k)$ play the role of a weight function. The precise alignment of these two sequences ensures that successive terms cancel in a structured manner, producing the remarkably simple closed-form expression for $L(n)$. This interpretation emphasizes that the balance of the tree is not merely numerical, but arises from a deeper algebraic interplay between weighted sequences and positional scaling.

3 Interpretation via Discrete-Time Linear Systems

The left-hand part of the tree can be interpreted as the response of a discrete-time linear system, in the sense of convolution and state-space realization [4]. But to really do this, it is necessary to reconcile the convolution up to the initial instant, that is

$$\begin{aligned}
L(n) = (b-1)&\sum_{k=1}^{n-1}(b-\kappa)b^{n-\kappa-1} + b - n - 1 \\
&\left((b-1)\sum_{\kappa=0}^{n-1} b^{n-1-\kappa}(b-\kappa) - b^{n-1}b\right) + (b-n-1) \\
&(b-1)\sum_{\kappa=0}^{n-1} b^{n-1-\kappa}(b-\kappa) - (b-1)(b^n-1) - n \\
&(b-1)b^n(-1) + (b-1)\sum_{\kappa=0}^{n-1} b^{n-1-\kappa}(b-\kappa) + b - n - 1
\end{aligned}$$

This yields directly the following state-based representation

$$\begin{cases} x_{\kappa+1} &= bx_\kappa + (b-\kappa) \\ y_\kappa &= (b-1)x_\kappa + (b-\kappa) - 1, \end{cases} \tag{1}$$

with $x_0 = -1$. A canonical form can be trivially obtained by augmenting the state by defining the linear system :

$$\hat{x} = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}, \quad A = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} b-1 \\ 1 \end{bmatrix}^T, \quad D = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

where $\tilde{x}_0 = -1$ is the required offset. The choice of B and D clearly filters out an input component along that coordinate. $\tilde{x}$. Of course we have $y_n = L(n) = R(n)$. When playing with integers and with $\kappa \leq b$ the linear system does produces the positional representations that reminds us the left-side of the tree. The appropriate affine transformation in the output y_n completes the picture! Interestingly, the interpretation of positional numbers representation assumes $n \leq b$. This is the reason why large trees are made with large bases.

Further spaces for enjoyment?
Are there further spaces for enjoyment when playing with real numbers as well as integers, and with the consequences of their ordering? If such space exists, however, one needs to look beyond. In fact, the beauty of the Christmas tree here also comes from the repetition of the number $b - 2$, namely at the tip of the tree. This is a consequence of ideas from the representation of integers.

References

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[4] Thomas Kailath. *Linear Systems.* Prentice-Hall, 1980.

[5] Donald E. Knuth. *The Art of Computer Programming, Volume 2: Seminumerical Algorithms.* Addison-Wesley, 3rd edition, 1997.